Midterm 2021

$\boxed{EX1}$

1) $dX_t = \alpha X_t dt + \beta X_t dW_t \quad \Rightarrow \quad X_t = X_0 \, e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}$

$F(X_t) = Y_t = X_t^{-1} \qquad F'(X_t) = -X_t^{-2} \qquad F''(X_t) = 2X_t^{-3}$

$dY_t = F'(X_t)\, dX_t + \frac{1}{2} F''(X_t)\, d[X_t] \qquad [X_t]_t = \beta^2 X_t^2 t$

$dY_t = -X_t^{-2} \cdot (\alpha X_t dt + \beta X_t dW_t) + X^{-3} \beta^2 X_t^2 dt$

$dY_t = -\alpha X_t^{-1} dt - \beta X_t^{-1} dW_t + \cancel{X_t^{-3}} X_t^{-1} \beta^2 dt =$

$dY_t = (\beta^2 - \alpha) \underbrace{X_t^{-1}}_{= Y_t} dt + (-\beta) \underbrace{X_t^{-1}}_{Y_t} dW_t$

$dY_t = \underbrace{(\beta^2 - \alpha)}_{\mu_1} Y_t dt + \underbrace{(-\beta)}_{\sigma_1} Y_t dW_t$

2) $Y_t = Y_0 + \underbrace{\int_0^t (\beta^2 - \alpha) Y_s ds}_{\substack{\text{finite first variation} \\ \Rightarrow \text{quadratic variation} = 0 \\ =: A_t}} + \underbrace{\int_0^t -\beta Y_s dW_s}_{=: M_t}$

$Y_t = Y_0 + A_t + M_t$

$[Y_t]_t = [M_t]_t$ because $[A_t] = 0$ as first variation of $[A]_t$ is finite

$[Y]_t = \int_0^t \beta^2 Y_s^2 \, d[W]_s = \int_0^t \beta^2 \frac{1}{X_s^2} ds$

3) Y_t is GBM where $\Rightarrow Y_t = Y_0 \cdot e^{-\beta \cdot W_t + \underbrace{[(\beta^2 - \alpha) - \frac{1}{2}\beta^2]}_{\frac{1}{2}\beta^2 - \alpha} \cdot t}$

$Y_0 = \frac{1}{x}$

$\boxed{Ex2}$

$$f_t + \alpha x f_x + \frac{1}{2}\beta^2 x^2 f_{xx} = rf \quad , \quad f(T,x) = \frac{1}{x}$$

•) $\mu(x) = \alpha x$

•) $\sigma^2(x) = \beta^2 x^2 \Rightarrow \sigma(x) = \beta x$

•) $r(x) = r$

•) $\phi(x) = \frac{1}{x}$

$X_t = \alpha \cancel{X_t} dt + \beta X_t dW_t$

$$f(t,x) = E_x\left(X_0^{-1} e^{-r(T-t)} \cdot X_{T-t}^{-1}\right) =$$

$$= X_0^{-1} e^{-r(T-t)} \cdot E_x\left(X_{T-t}^{-1}\right) = X_0^{-1} e^{-r\cdot(T-t)} \cdot E\left(e^{-\beta\cdot W_{T-t} + \left(\frac{1}{2}\beta^2 - \alpha\right)\cdot(T-t)}\right)$$

$$= X_0^{-1} e^{-r\cdot(T-t)} \cdot E\left(e^{-\beta W_{T-t} + \left(\frac{1}{2}\beta^2 - \alpha\right)\cdot(T-t)}\right) =$$

$$\cancel{-\beta \underbrace{W_{T-t}}_{\sim N(0,(T-t))} + \left(\frac{1}{2}\beta^2 - \alpha\right)\cdot(T-t) \sim N\left(\left[\frac{1}{2}\beta^2 - \alpha\right](T-t), \beta^2\cdot(T-t)\right)}$$

$$= X_0^{-1} e^{-r\cdot(T-t)} e^{\left(\frac{1}{2}\beta^2 - \alpha\right)(T-t)} \cdot E\left(e^{-\beta W_{T-t}}\right)$$

$$-\beta W_{T-t} \sim N(0, \beta^2(T-t))$$

$$E\left(e^{-\beta W_{T-t}}\right) = \frac{1}{2}\beta^2\cdot(T-t)$$

$$= X_0^{-1} e^{-r\cdot(T-t)} e^{\left(\frac{1}{2}\beta^2 - \alpha\right)\cdot(T-t)} e^{\frac{1}{2}\beta^2\cdot(T-t)} = X_0^{-1} e^{(T-t)\cdot(-r+\beta^2-\alpha)}$$

$$f(t,x) = X_0^{-1} e^{(T-t)(-r+\beta^2-\alpha)}$$

$$f(T,x) = X_0^{-1} e^{(T-T)\cdots} = \frac{1}{x} \checkmark$$

$$f_t = f(t,x)\cdot(r-\beta^2+\alpha)$$

$$f_x = -\frac{1}{x}\cdot f(t,x) = -\frac{1}{x^2}\cdots$$

$$f_{xx} = 2\frac{1}{x^2}\cdot f(t,x)$$

$$f(t,x)\cdot(r-\beta^2+\alpha) - \alpha \cancel{x}\frac{1}{\cancel{x}} f(t,x) + \cancel{\frac{1}{2}}\beta^2 \cancel{x^2}\cancel{2}\cdot\frac{1}{\cancel{x^2}} f(t,x) =$$

$$= f(t,x)\cdot(r-\beta^2+\alpha) - \alpha f(t,x) + \beta^2 f(t,x)$$

$$= f(t,x)\cdot(r - \cancel{\beta^2} + \cancel{\alpha} - \cancel{\alpha} + \cancel{\beta^2}) =$$

$$= f\cdot r \checkmark$$

Ex3 $\nearrow$ stopping time

τ on some filtered prob. space (Ω, F, P)

$\{F_t, t \geq 0\}$, $s > 0$ $\quad \tilde{\tau} = \tau + s \Rightarrow \tilde{\tau} - s = \tau$

Definition of stopping time

$\forall t > 0$, $\{\tau \leq t\} \in F_t$

$\{\tilde{\tau} = \tau + s \leq t\} = \{\tau \leq t - s\} \in F_t$

2 Cases $\quad t \geq s \Rightarrow \{\tilde{\tau} \leq t\} = \{\tau \leq \underbrace{t-s}_{\geq 0}\} \in F_{t-s} \in F_t$

$t < s \Rightarrow \{\tilde{\tau} \leq t\} = \{\tau \leq t - s\} = \emptyset \in F_t$

$\Rightarrow \tilde{\tau}$ is stopping time